bilinear form

$$a(v, w) = \int_\Omega \nabla v \cdot A \nabla w \, dx$$

$H = H_0^1(\Omega)$, $\ell(v) = \int_\Omega f v \, dx$

multiscale problem

$u \in H$: $a(u, v) = \ell(v) \; \forall v \in H$

ellipticity:

$$\gamma \|v\|_1^2 \leq a(v, v)$$

$$|a(v, w)| \leq \Gamma \|v\|_1 \|w\|_1$$

$\Leftrightarrow$ existence, uniqueness, stability of solutions

condition number Γ/γ

depends on global contrast M/δ

linear projection operator

from now on let

$$\Pi : H \to S_H$$

denote a linear projection

a) $\Pi(\alpha v + \beta w) = \alpha \Pi v + \beta \Pi w$

$$\Pi^2 = \Pi$$

with the properties

b) $|\Pi v|_1 \leq c_1 |v|_1$ (boundedness)

c) $\| H^{-1}(v - \Pi(v)) \|_0$

$$\leq c_2 |v|_1$$

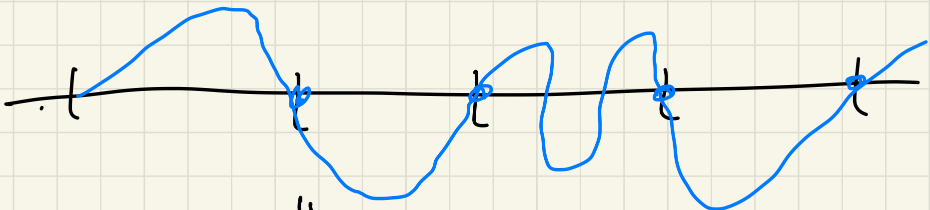
(approximation property)

orthogonal decomposition of H

(Målqvist & Peterseim 14)

$$\mathcal{V} = \ker \Pi = \{ v \in H \mid \Pi v = 0 \}$$

example: $d = 1$, $\Pi = I_H$



$\mathcal{V}$ is "high-frequency"

a-orthogonal projection

$$C : H \longrightarrow \mathcal{V}$$

$$Cw \in \mathcal{V} : \; a(Cw, v) = a(w, v) \quad \forall v \in H$$

a-orthogonal complement of $\mathcal{V}$

$$\mathcal{W} := (I - C)\, H$$

Lemma 10

a) $(I-C)v = (I-C)\Pi v \quad \forall v \in H$

b) $W := (I-C)S_H = (I-C)H$

c) $\dim W = \dim S_H$

d) $W = \operatorname{span}\{(I-C)z_p \mid p \in \mathcal{N}\}$

Proof:

a) $(I-C)v = (I-C)(\Pi v + \underbrace{(I-\Pi)v}_{\in V})$

$= (I-C)\Pi v + (I-C)\overbrace{\cancel{C(I-\Pi)}}^{=0}v$

b) $W = (I-C)H = (I-C)\Pi H = (I-C)S_H$

c) we show $\left((I-C)|_{S_H}\right)^{-1} = \Pi|_W$, i.e.

(i) $(I-C)\Pi w = w \quad \forall w \in W$

(ii) $\Pi(I-C)v = v \quad \forall v \in S_H$

(i) $(I - C)\Pi w = w \quad \forall w \in W$

$w = (I - C) v \in W$ for some $v \in S_H$

$(I - C)\Pi w = (I - C)\Pi (I - C) v$

$= (I - C)(\Pi v - \Pi C v)$

$= (I - C) \quad v = w$

(ii) $\Pi (I - C) v \quad = v \quad \forall v \in S_H$

$\Pi (I - C) v = \Pi v - \Pi C v$

$= v$

d) $\quad W = (I - C) S_H$

$= \{ w \mid (I - C) \sum_p v_p \lambda_p, \ v_p \in \mathbb{R} \}$

$= \{ w \mid \sum_p v_p (I - C) \lambda_p, v_p \in \mathbb{R} \}$

note: $\Pi (I - C) \lambda_p = \lambda_p$

modified finite element approx.

$w = P_{\mathcal{W}} u \in \mathcal{W}$:

$$a(w, v) = (f, v) \quad \forall v \in \mathcal{W}$$
$$= a(u, v)$$

Proposition 11

(Peterseim, Hughes & Sangalli)

a) $w = (\underline{I} - C) u \Leftrightarrow u - w = Cu$

b) $w = (\underline{I} - C) \Pi u$

Proof: Let $v = (\underline{I} - C) v_H \in \mathcal{W}$, $v_H \in H$

a) $a((\underline{I} - C) u, v) =$

$= a(u, v) - a(Cu, v)$

$= l(v) - \underbrace{\cancel{a(u, Cv)}}_{= 0}$

$= l(v) \quad \forall v \in \mathcal{W} = (\underline{I} - C) H$

b) Lemma 10 a) $\blacksquare$

Proposition 12

Discretization error estimate

$$\| u - w \| \leq c_2 \frac{1}{\rho^{1/2}} \| H f \|_0$$

Proof:

$$\| u - w \|^2 = a(Cu, Cu)$$

$$= a(u, Cu - \Pi Cu)$$

$$= (Hf, H^{-1}(I - \Pi) Cu)$$

$$\leq \| Hf \|_0 \, \| H^{-1}(I - \Pi) Cu \|_0$$

$$\leq \| Hf \|_0 \, c_2 \, | Cu |_1$$

$$\leq \| Hf \| \, c_2 \frac{1}{\rho^{1/2}} \| u - w \|$$

$\boxtimes$

modified finite element basis

basis of $\mathcal{W}$: $\lambda_p - C\lambda_p,\ p \in \mathcal{N}_H$

orthogonalization

$C\lambda_p \in \mathcal{V} = \ker \Pi$:

$$a(C\lambda_p, v) = a(\lambda_p, v) \quad \forall v \in \mathcal{V}$$

drawback: $\operatorname{supp} C\lambda_p = \Omega$

$\Leftrightarrow$ fully occupied stiffness matrix $A_{\mathcal{W}}$

$\Leftrightarrow$ complexity of $A_{\mathcal{W}} v$: $\mathcal{O}(|\mathcal{N}_H|^2)$

Remark:

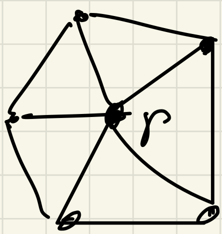
(i) comp. feasible version: $\mathcal{V} \cap S_h$

(ii) global support of $\lambda_p - C\lambda_p$

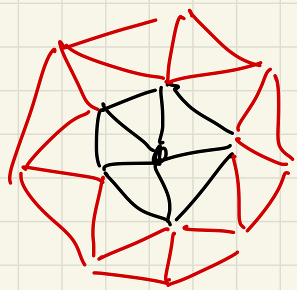
goal: Localized orthogonal decomposition (LOD)

1. idea: prescribe support by preserving the a priori error estimate (Målqvist & Peterseim 14)

patches of order ℓ: $\omega_{p,\ell}$



$\omega_{p,1}$



$\omega_{p,2}$

Målqvist & Peterseim:

$$\ell \gtrsim |\log H| \Rightarrow \|u - w\| \lesssim H$$

drawbacks:

- a priori selection of ℓ
- saddle point problems: $v \in V$, $\operatorname{supp} v \subset \omega_{p,\ell}$
- complicated proof

2. idea: localization by subspace correction (approximate orthogonalization)

Theorem 13:

Let $C_\ell : H_0^1(\Omega) \to V$ satisfy

$$\| C v - C_\ell v \| \le \varrho^\ell \| C v \|$$

$$C_0 v = 0, \quad 0 < \varrho < 1$$

and let

$$W_\ell = \operatorname{span} \{ \lambda_p - C_\ell \lambda_p \mid p \in N_H \}$$

Then $w_\ell = P_{W_\ell} u$ satisfies

$$\| u - w_\ell \| \le (1 + \varrho^\ell) \| u - w \| + \varrho^\ell \| u - \Pi u \|$$

Proof:

Note that

$$\Pi u - C_\ell \Pi u = (I - C_\ell)\,\Pi u$$

$$= \sum_{p \in N_H} (\Pi u)(p)\,(I - C_\ell)\lambda_p \in W_\ell$$

$$\| u - w_\ell \| \le \inf_{v \in W_\ell} \| u - v \|$$

$$\le \| u - (\Pi u - C_\ell \Pi u) \|$$

$$\le \| u - w + \cancel{\Pi u} - C \Pi u - \cancel{\Pi u} + C_\ell \Pi u \|$$

$$\le \| u - w \| + \| (C - C_\ell)\, \Pi u \|$$

$$\le \| u - w \| + g^\ell \, \| C\, \Pi u \|$$

$$\le \| u - w \| + g^\ell \, \| \Pi u - w \|$$

$$\leq \|u - w\| + \rho^\ell \|\Pi u - w\|$$

$$\leq \|u - w\| + \rho^\ell (\|\Pi u - u\| + \|u - w\|)$$

$$= (1 + \rho^\ell)\|u - w\| + \rho^\ell \|\Pi u - u\|$$

$\square$

As an example for C_ℓ we consider the preconditioned Richardson iteration

$$(R) \quad C_{\ell+1} v = C_\ell v + \omega T (v - C_\ell v)$$

with $C_0 v = 0$

preconditioner:

$$T = P_1 + P_2 + \dots + P_n$$

with $P_i = P_{V_i}$ and

and the decomposition

$$V = V_1 + \dots + V_n$$

$$V_i = (v - \Pi v)\, H_0^1(\omega_{p_i,1})$$