modified finite element basis

basis of W: $\lambda_p - C\lambda_p$, $p \in N_H$

orthogonalization

$C\lambda_p \in V = \ker \Pi$:

$$a(C\lambda_p, v) = a(\lambda_p, v) \quad \forall v \in V$$

drawback: $\operatorname{supp} C\lambda_p = \Omega$

$\Leftrightarrow$ fully occupied stiffness matrix A_W

$\Leftrightarrow$ complexity of $A_W v$: $O(|N_H|^2)$

Remark:

(i) comp. feasible version: $V \cap S_h$

(ii) global support of $\lambda_p - C\lambda_p$

2. idea: localization by subspace correction (approximate orthogonalization)

Theorem 13:

Let $C_\ell : H_0^1(\Omega) \to V$ satisfy

$$\| C v - C_\ell v \| \leq \varrho^\ell \| C v \|$$

$$C_0 v = 0, \quad 0 < \varrho < 1$$

and let

$$W_\ell = \operatorname{span} \{ \lambda_p - C_\ell \lambda_p \mid p \in N_H \}$$

Then $w_\ell = P_{W_\ell} u$ satisfies

$$\| u - w_\ell \| \leq (1 + \varrho^\ell) \| u - w \| + \varrho^\ell \| u - \Pi u \|$$

As an example for C_ℓ we consider the preconditioned Richardson iteration

$$(R) \quad C_{\ell+1} v = C_\ell v + \omega T (v - C_\ell v)$$

with $C_0 v = 0$ and

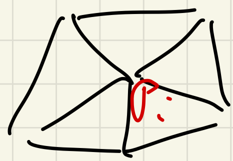
$$T = P_1 + \dots + P_n$$

$$P_i w \in V_i : a(P_i w, v) = a(w, v) \ \forall v \in V_i$$

and $V = \ker \Pi = V_1 + \dots + V_n$

$$V_i = (I - \Pi) H_0^1 (\omega_{p_i})$$

$\omega_{p_i} = \operatorname{supp} \lambda_{p_i}$



ω_{p_i}

Proposition 14:

Let Π be the Clément quasiinterpolation. Then

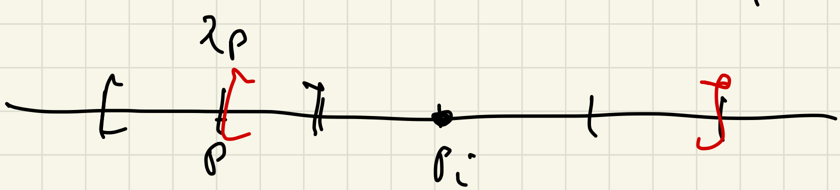
$$\operatorname{supp}(\lambda_p - C_\ell \lambda_p) \subset \omega_{p,3\ell+1}$$

Proof:

$\operatorname{supp} \Pi v \subset \omega_{p,2}$ for $v \in H_0^1(\omega_p)$

$\Leftrightarrow \quad v \in V_i \Rightarrow \operatorname{supp} v \subset \omega_{p_i,2}$

$\operatorname{supp}(\lambda_p - C_0 \lambda_p) = \omega_{p,1}$



$\operatorname{supp}(\lambda_p - C_1 \lambda_p) = \omega_{p,1+3}$

by induction: $\operatorname{supp}(\lambda_p - C_\ell \lambda_p)$

$= \omega_{p,3\ell+1}$

Remark:

Let $\Pi = I_H$ (nodal interpolation)

Then $\operatorname{supp} \Pi v \subset \omega_p \; \forall v \in H_0^1(\omega_p)$

$\Leftrightarrow \operatorname{supp}(\lambda_p - C_\ell \lambda_p) \subset \omega_{p,\ell}$

but I_H is restricted to $d = 1$

Remark:

no saddle point problems
occur in the iteration
(basis of $(I - \Pi) S_H(\omega_p)$ in
comp. feasible versions)

Lemma 15

$$\|\nabla \lambda_p\|_{\infty, t} = \max_{x \in t} |\nabla \lambda_p(x)| \lesssim H_t^{-1}$$

Lemma 16

(V1) $\forall v \in V \ \exists v_i \in V_i : v = v_1 + \dots + v_n$

$$\sum_{i=1}^{n} \|v_i\|^2 \leq K_1 \|v\|^2$$

(V2) $\forall v \in V \ \forall v_i \in V_i : v = v_1 + \dots + v_n$

$$\|v\|^2 \leq K_2 \sum_{i=1}^{n} \|v_i\|^2$$

with K_1, K_2 independent of H

(contrast M/δ, shape regularity)

Proof:

(V1) Let $v \in V$, $v_i = (\underline{I} - \Pi)(\lambda_{p_i} v)$

Then $\sum_{i=1}^{n} v_i = (I - \Pi)\underbrace{\left(\sum_{i=1}^{n} \lambda_{p_i}\right)}_{=1} v = v$

$$\sum_{i=1}^{n} \| v_i \|^2 \overset{\text{ellipt.}}{\lesssim} \sum_{i=1}^{n} \| v_i \|_1^2$$

$$\overset{\text{loc. Princ.}}{\lesssim} \sum_{i=1}^{n} | v_i |_1^2 = \sum_{i=1}^{n} | (I - \Pi) \lambda_{p_i} v |_1^2$$

$$\leq \sum_{i=1}^{n} (1 + c_1)^2 | \lambda_{p_i} v |_1^2$$

$$\lesssim 2 \sum_{i=1}^{n} \| |\nabla \lambda_{p_i}| v \|_0^2 + \| \lambda_{p_i} |\nabla v| \|_0^2$$

$$\lesssim \sum_{t \in T_H} \sum_{p_i \in N(t)} \| \nabla \lambda_{p_i} \|_{\infty, t}^2 \| v \|_{0,t}^2$$

$$+ \| \lambda_{p_i} \|_{\infty, t}^2 | v |_{1,t}^2$$

$$\lesssim \sum_{t \in \mathcal{T}_H} \sum_{p_i \in \mathcal{N}(t)} \|\nabla \lambda_{p_i}\|^2_{\infty,t} \|v\|^2_{0,t}$$

$$+ \|\lambda_{p_i}\|^2_{\infty,t} |v|^2_{1,t}$$

$$\lesssim \sum_{t \in \mathcal{T}_H} (d+1)\, H_t^{-2} \|v\|^2_{0,t} + |v|^2_{1,t}$$

$$\lesssim \sum_{t \in \mathcal{T}_H} \|H^{-1} v\|^2_{0,t} + |v|^2_{1,t}$$

$$= \|H^{-1} v\|^2_0 + |v|^2_1$$

$$= \|H^{-1}(v - \Pi(v))\|^2_0 + |v|^2_1$$

approx.

$$\lesssim |v|^2_1$$

(V2) Let $v \in V$, $v_i \in V_i : v = v_1 + \dots + v_n$

$$\| v \|^2 = \sum_{i,j=1}^{n} a(v_i, v_j)$$

$$\leq \sum_{i,j=1}^{n} \varepsilon_{ij} \| v_i \| \| v_j \| = \underline{v} \cdot \mathcal{E} \underline{v}$$

$$\leq K_2 | \underline{v} |^2 = K_2 \sum_{j=1}^{n} \| v_j \|^2$$

with $K_2 \leq$ const. (shape reg.)

$$\mathcal{E} = (\varepsilon_{ij}), \quad \varepsilon_{ij} = \begin{cases} 1 & \omega_{p_i,2} \cap \omega_{p_j,2} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Theorem 17

The preconditioned Richardson iteration (R) converges for $\omega = 1/K_2$ with convergence rate

$$\rho = 1 - 2/(K_1 K_2 + 1) < 1$$

Proof: Corollary 7

Corollary 18

Required accuracy (cf. Theorem 13)

(S) $\rho^{l} \| u - \Pi u \| \lesssim H$ (stopping crit.)

$\Leftrightarrow \; l \lesssim |\log H|$

(S) can be controlled a posteriori based on error estimation

computationally feasible variant

Let S_h be a space of piecewise linear finite elements that resolves all scales

special case: $A(x) = \alpha(x, \frac{x}{\varepsilon})\, I$

triangulation $\mathcal{T}_h$ with $h = H\varepsilon$

$\Leftrightarrow \quad \| u - u_h \| \lesssim \frac{1}{\varepsilon} h = H$

Replace $H_0^1(\Omega)$ by S_h and

$V_i = \{ v - \Pi[v] \mid v \in S_h \cap H_0^1(\omega_{P_i}) \}$

a priori error estimate:

$$\| u - u_H \| \leq \| u - u_h \| + \| u_h - u_H \|$$

$$\lesssim H + (1+\varrho^l) \| H f \| + \varrho^l \| u_h - \Pi u_h \|$$

Proofs:

direct analogues of Proposition 12 and Theorem 13 (literally the same)

1.3.2 Multigrid methods for highly oscillating coefficients (Xu 92, Kh & Yserentant 2016)

goal: iterative approximation of $u_h \in S_h$ by solving one global problem and local subproblems.

two-level subspace decomposition:

$$V = H_0^1(\Omega)$$

$$V = V_0 + V_1 + \ldots + V_n$$

$$V_0 = S_H$$

$$V_i = H_0^1(\omega_{P_i})$$

preconditioner:

$$T = P_0 + P_1 + P_2 + \dots + P_n$$

Theorem 1a

(V1) $\forall v \in V \ \exists v_i \in V_i : v = v_0 + \dots + v_n$

$$\sum_{i=0}^{n} \|v_i\|^2 \leq K_1 \|v\|^2$$

(V2) $\forall v \in V \ \forall v_i \in V_i : v = v_0 + \dots + v_n$

$$\|v\|^2 \leq K_2 \sum_{i=0}^{n} \|v_i\|^2$$

Proof:

(V1) Let $\Pi : H_0^1(\Omega) \to S_H$

be a projection with properties

$$|\Pi v|_1 \lesssim |v|_1,$$

$$\| H^{-1}(v - \Pi v)\|_0 \lesssim |v|_1$$

Set $v_0 = \Pi v \in S_H = V_0$

$v_i = \lambda_{p_i}(v - \Pi v) \in V_i$

Then

$v_0 + v_1 + \dots + v_n =$

$$\Pi v + \underbrace{\left(\sum_{i=1}^{n} \lambda_{p_i}\right)}_{=1}(v - \Pi v) = v$$

$$\sum_{i=0}^{n} \| v_i \|^2 = \| \Pi v \|^2 + \sum_{i=1}^{n} \| \chi_{\rho_i} (v - \Pi v) \|^2$$

$$\lesssim | \Pi v |_1^2 + \sum_{i=1}^{n} | \chi_{\rho_i} (v - \Pi v) |_1^2$$

$$\lesssim | v |_1^2 + \| H^{-1} (v - \Pi v) \|_0^2$$

$$+ | v - \Pi v |_1^2$$

$$\lesssim | v |_1^2$$

comparison of cost

scale separation $A = \alpha(x/\varepsilon)\, I$

$\Rightarrow$ $h = H\varepsilon$ $\quad \|u - u_h\| \lesssim H$

multigrid:

iteration steps : $|\ln H|$

V_0 : size $1/H \le H/h$ as $\frac{h}{H} = \varepsilon \le H$

V_i : size H/h

cost per lin. system $(\text{size})^\beta$ $\beta \geq 1$

overall cost:

$$\text{cost}_{MG} \lesssim |\log H| \left(H/h\right)^\beta$$

multiscale finite elements:

V_0 : size $1/H \leq H/h$

V_i : size $H |\log H|^2 / h$

overall cost

$$\mathrm{cost}_{\mathrm{MS}} \lesssim \left(\frac{H |\log H|^2}{h} \right)^\beta$$

$$\gtrsim \mathrm{cost}_{\mathrm{reg}} (\log H)^{2\beta - 1}$$

upshot:
multiscale FE only
if multiscale FE basis
can be reused.