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Exercise 1 for the lecture

Numerical Mathematics II

WS 2021/2022

http://numerik.mi.fu-berlin.de/wiki/WS_2021/NumericsII.php

Due: Tutorial on November 2, 2021

Problem 1

Consider the ODE

$$x'(t) = f(x(t)), \qquad 0 < t \leq T.$$

Assume that f is Lipschitz with constant $L > 0$ and let ϕ^t be the associated flow operator. Consider a numerical approximation scheme with discrete flow ψ^τ. Let ψ^τ be stable with a constant $\gamma > 0$ and consistent with order p.

Show that the associated one step method

$$x_{k+1} = \psi^\tau x_k, \qquad k = 0, \dots, n-1$$

is convergent with order p.

Hint: Show by induction that

$$\|x(t_k) - x_k\| \leq e^{t_k \gamma L} \sum_{i=0}^{k-1} \|\varepsilon_i\|$$

holds for $k = 1, \dots, n-1$ with

$$\varepsilon_k = \phi^\tau x_k - \psi^\tau x_k, \quad k = 0, \dots, n-1.$$

Problem 2

Let $\lambda \in \mathbb{R}$. For the linear differential equation

$$x'(t) = \lambda x(t), \qquad 0 < t \leq T,$$

we consider the following numerical approximation scheme:

$$\begin{aligned} \tilde{x}_{k+1} &= x_k + \tau \lambda x_k, \\ x_{k+1} &= x_k + \frac{\tau}{2} \left(\lambda x_k + \lambda \tilde{x}_{k+1} \right). \end{aligned}$$

a) Write down explicitly the flow operator ϕ^t of the ODE and the discrete flow operator ψ^τ of the numerical scheme.

b) Show that ψ^τ is consistent with order $p = 2$.

c) Show that ψ^τ is stable.

Problem 3

a) Implement a function

```
exp_euler(x0, f, tau, n)
```

which returns a vector containing `n` steps of the explicit Euler method $x_{k+1} = x_k + \tau f(x_k)$ approximating the IVP $x'(t) = f(x(t))$ with initial value `x0`.

b) Analogously to a), implement a function

```
heun(x0, f, tau, n)
```

which returns a vector containing `n` steps of Heun's method which is the numerical scheme from the previous exercise.

c) Consider the IVP

$$\begin{aligned} x'(t) &= -x, \qquad 0 < t \leq T = 1.5, \\ x(0) &= 1, \end{aligned}$$

which has the unique solution $x(t) = e^{-t}$. Apply both methods to this problem using $n = 2, ..., 1000$ and $\tau = \frac{T}{n}$. Compute the respective errors

$$\max_{k=0,\dots,n} |x(t_k) - x_k|, \qquad t_k = k\tau,$$

and compare them by plotting the errors over n using a suitable scaling. Commend on the convergence rates.