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Exercise 2 for the lecture

Numerical Mathematics II

WS 2021/2022

http://numerik.mi.fu-berlin.de/wiki/WS_2021/NumericsII.php

Due: Tutorial on November 9, 2021

Problem 1

Show that the matrix exponential has the following properties

a)
$$e^{t(A+B)} = e^{tA}e^{tB}, \quad \forall A, B \in \mathbb{R}^{d\times d} \text{ with } AB = BA$$

b)
$$A = \operatorname{blockdiag}(A_1, \dots, A_k) \Rightarrow e^{tA} = \operatorname{blockdiag}(e^{tA_1}, \dots, e^{tA_k})$$

c)
$$e^{\alpha I} = e^{\alpha} I, \quad \alpha \in \mathbb{R},\, I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathbb{R}^{d\times d}.$$

Problem 2

Let Φ^t be a flow on $\mathbb{R}^d$. Let $x_0 \in \mathbb{R}^d$ be stable in the following sense:

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in \mathbb{R}^d \;\; (|x - x_0| < \delta \quad \Rightarrow \quad \forall t > 0 \;\; |\Phi^t(x) - \Phi^t(x_0)| < \varepsilon).$$

Prove or disprove: x_0 is a fixed point of the flow.

Problem 3
Consider the one-dimensional ODEs

a) $x' = x - x^2$

b) $x' = -x + 4x^3 - x^5$

c) $x' = \begin{cases} 0, & x = 0 \\ -x^3 \sin(\frac{1}{x}), & x \neq 0. \end{cases}$

Find all fixed points and if possible discuss their stability based on the statements from the lecture.