Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Ralf Kornhuber, Lasse Hinrichsen-Bischoff

Exercise 7 for the lecture

Numerical Mathematics II

WS 2021/2022

http://numerik.mi.fu-berlin.de/wiki/WS_2021/NumericsII.php

Due: 11:59pm on Monday, December 13, 2021

Problem 1

Let $E : \mathbb{R}^n \to \mathbb{R}$ be a convex functional and consider the associated *gradient flow*

$$x'(t) = -\nabla E(x(t)), \qquad x(0) = x_0, \tag{1}$$

where $\nabla E(x(t)) \in \mathbb{R}^n$ is the gradient of E at $x(t)$.

a) Show that $E(x(t)) \leq E(x_0)$ for all $t > 0$. Show then that even $E(x(t)) < E(x_0)$ if $\nabla E(x_0) \neq 0$.

b) Show that $x^* \in \mathbb{R}^n$ is a fixed point of (1) if and only if x^* is a minimum of E. Furthermore, show that each isolated fixed point of (1) is stable.

c) Assume that E is strictly convex. Show that each fixed point of (1) is asymptotically stable.

d) Let E be strictly convex and coercive. Then $x^* = \lim_{t\to\infty} x(t)$ is a fixed point of the gradient flow for all x_0.

Problem 2

For a linear problem $x' = Ax$, the Rosenbrock-Wanner method $\begin{array}{c|c} \mathbb{B} & \mathbb{A} \\ \hline & b^T \end{array}$ is equivalent to the implicit Runge-Kutta method $\begin{array}{c|c} & \mathbb{B} \\ \hline & b^T \end{array}$. Thus the stability function is given by

$$R(z) = 1 + zb^T(1 - z\mathbb{B})^{-1}e, \quad e = (1, \ldots, 1)^T \in \mathbb{R}^s. \tag{2}$$

Deduce from (2), that the condition $\sum_{i=1}^{s} b_i = 1$ is sufficient for consistency (order p=1).

Problem 3
Consider the linear scalar ODE

$$x'(t) = \lambda x(t), \qquad x(0) = x_0.$$

a) Use the implicit Euler method $\Psi_*^\tau x = x + \tau\lambda(\Psi_*^\tau x)$ to implement a function that does one step of the extrapolation method using timestep `tau`:

```
extrapolation(x, lamda, tau, m)
```

I. e. your function should return $\Psi^\tau x = T_{mm}$ (using the notation from the lecture notes).

b) Let $\lambda = -1$ and $x_0 = 1$. For each of $m \in \{2, \dots, 5\}$, compute the error

$$|\exp(-\tau)x_0 - \Psi^\tau x_0|$$

for $\tau \in \{10^{-i} : i = 1, \dots, 10\}$. Plot these errors over τ using a suitable scaling. Do your computations confirm the consistency order m?

Hint: You can use any interpolation method you want. However, the Aitken–Neville scheme might be particularly useful since you only need the point value T_{mm}.