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Exercise 10 for the lecture

Numerical Mathematics II

WS 2021/2022

http://numerik.mi.fu-berlin.de/wiki/WS_2021/NumericsII.php

Due: 11:59pm on Monday, January 17, 2022

Problem 1

Consider the initial value problem

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}, \qquad \begin{pmatrix} q \\ p \end{pmatrix}(0) = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}. \tag{1}$$

Let the flow $\Phi^t x_0$ be the solution of (1) with initial condition $x_0 = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$. Show that the flow map Φ^t conserves the Hamiltonian function

$$H(q,p) = \frac{1}{2}(q^2 + p^2)$$

i.e. for all times t we have

$$H(\Phi^t x_0) = H(x_0).$$

Furthermore, show that neither the explicit nor implicit Euler methods Ψ^τ conserve H i.e. $H(\Psi^\tau x) \neq H(x)$, for any $x \in \mathbb{R}^2$.

Problem 2

Let $\varphi : U \to V$ be a change of coordinates such that φ and φ^{-1} are continuously differentiable functions. If φ is a symplectic transformation, then it preserves the Hamiltonian character of an ODE in the following sense:
The Hamiltonian system $y' = -J\nabla H(y)$ in the new coordinates $z = \varphi(y)$ becomes

$$z' = -J\nabla K(z) \quad \text{where } K(z) = H(\varphi^{-1}(z)). \tag{2}$$

Conversely, if φ transforms every Hamiltonian system to another Hamiltonian system via (2), then φ is symplectic.

Problem 3

Consider the pendulum equation in polar coordinates

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \frac{1}{m} p \\ -m\frac{g}{r_0}\cos q \end{pmatrix} \qquad \begin{pmatrix} q \\ p \end{pmatrix}(0) = \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}$$

where q, g, m, and r_0 denote the angle, the gravity, the mass and the radius, respectively.

a) Implement the symplectic Euler method for this equation in `matlab` as `function [p, q, t] = SymplecticEuler(m, g, r0, p0, q0, I, tau)`, where `(m, g, r0)`, `(p0, q0)`, `I`, and `tau` denote the problem parameters, the initial values, the time interval and the step size, respectively.

b) Test your program with the radius $r_0 = 10cm$, the mass $m = 100g$, and the gravity of the moon. Use the time interval $[0s, 20s]$ with the initial value $p_0 = 0\frac{kgm}{s}$, $q_0 = 0m$ for various time step sizes.

c) Plot the solution in the phase space, the solution in Euclidean coordinates, and the associated Hamiltonian H.

d) Show that the radius $r = \sqrt{x_1^2 + x_2^2}$ is preserved if $g > 0$.