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Exercise 11 for the lecture

Numerical Mathematics II

WS 2021/2022

http://numerik.mi.fu-berlin.de/wiki/WS_2021/NumericsII.php

Due: 11:59pm on Monday, January 24, 2022

Problem 1

a) Assume that $A \in \mathbb{R}^{n \times n}$ is strongly diagonal dominant, i.e.

$$\sum_{j=1, j \neq i}^{n} |a_{ij}| < |a_{ii}|, \qquad \forall i = 1, \ldots, n. \tag{1}$$

Prove that the Jacobi iteration is globally convergent.

b) Is (1) a necessary condition for global convergence?

c) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, with $\|A\| < 1$. Show that for spectral radius of A it holds $\rho(A) < 1$.

d) Let $A \in \mathbb{R}^{n \times n}$ be symmetric with $\rho(A) < 1$. Show that for some norm on $\mathbb{R}^n$ it holds that $\|A\| < 1$.

Problem 2

Let $A, B \in \mathbb{R}^{n \times n}$ symmetric. Show that

$$\kappa(B^{-1}A) \leq \frac{\mu_1}{\mu_0}$$

holds, if $B \in \mathbb{R}^{n \times n}$ is a preconditioner satisfying

$$\mu_0 (Bx, x) \leq (Ax, x) \leq \mu_1 (Bx, x) \quad \forall x \in \mathbb{R}^n$$

for some $0 < \mu_0, \mu_1 \in \mathbb{R}$.

Problem 3
Let $A \in \mathbb{R}^{n,n}$ be symmetric positive definite and $b \in \mathbb{R}^n$.

a) Show that there is a strictly convex functional $J : \mathbb{R}^n \to \mathbb{R}$ such that the solution of the linear system $Ax = b$ is the minimizer of J.

b) Show that the Richardson iteration

$$x_{k+1} = x_k + \omega(b - Ax_k) \tag{2}$$

with $\omega \in \mathbb{R}$ is equivalent to the explicit Euler method for the gradient flow associated with J.

c) Show that there is an $\omega > 0$ such that the Richardson iteration converges to the solution of the linear system without using the results of Chapter 4 on numerical linear algebra.

d) Why is using an implicit Runge-Kutta method for the gradient flow not a good idea to solve the linear system ?